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$1/N$ corrections in Calogero-type models using the collective-field method

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Abstract. The collective field is applied to treat higher-order terms in the $1/N$ expansion of the ground-state energy in the Calogero model and related O(N), U(N) and Sp(N) invariant matrix models. All these models share the common structure in the large-N limit and have a unified description in terms of collective fields.

1. Introduction

The Calogero model (Calogero 1969) is one of the few exactly solvable one-dimensional quantum mechanical models of N-particle systems (Perelomov 1980). On the other hand, this model is related to O(N), U(N) and Sp(N) invariant matrix models which have been investigated in the past few years in connection with the large-N expansion of simplified models of quantum chromodynamics (Jevicki and Sakita 1980). There were also attempts to treat the Sp(N) invariant model using the large-N method with collective fields (de Carvallo and Fateev 1981). It was shown that the treatment of the Sp(N) model needs a careful analysis because of the specific singular interaction of the Calogero type (Andrić and Jevicki 1983).

In the present paper we study the Calogero model and develop the collective-field method to treat higher-order corrections in the $1/N$ expansion. Knowing the wavefunctional in the leading order determines the next-to-leading term in energy. We obtain the following results. Firstly, we show that the Calogero-type model for an N-particle system possesses semiclassical behaviour in the large-N limit. Secondly, the corrections to the leading behaviour in N can be expressed in terms of collective fields and their correlations.

2. Calogero interaction and related one-matrix models

We start from the N-particle Hamiltonian with the Calogero interaction

$$H=\frac{1}{2}\sum_{i=1}^{N} p_i^2+\frac{g}{2}\sum_{i\neq j}^{N}\frac{1}{(x_i-x_j)^2}+\frac{\omega^2}{8}\sum_{i\neq j}^{N}(x_i-x_j)^2. \tag{1}$$

Because of the singularity of the Hamiltonian for $x_i=x_j$, the wavefunction ought to have a prefactor which will vanish for $x_i=x_j$. We shall extract this prefactor in the form

$$\psi(x_1,\ldots,x_N)=\prod_{i<j}^{N}(x_i-x_j)^{\lambda}\Phi(x_1,\ldots,x_N)\equiv\Delta(x_i-x_j)\Phi(x_1,\ldots,x_N). \tag{2}$$

Using the identity

$$\sum_i^N \left(\sum_{j\neq i}^N \frac{1}{x_i - x_j} \right)^2 = \sum_{i\neq j}^N \frac{1}{(x_i - x_j)^2}$$

and choosing $\lambda \geqslant 0$ such that $\lambda(\lambda-1) = g$, we obtain

$$\left(-\frac{1}{2} \frac{1}{\Delta^{2\lambda}} \sum_{i=1}^N \frac{\partial}{\partial x_i} \Delta^{2\lambda} \frac{\partial}{\partial x_i} + \frac{\omega^2}{8} \sum_{i\neq j}^N (x_i - x_j)^2 \right) \Phi = E\Phi. \qquad (3)$$

Owing to the permutational symmetry of the Hamiltonian, we expect that the ground state will also keep this symmetry. We must therefore introduce a collective field which has permutation symmetry:

$$\rho(x) = \sum_{i=1}^N \delta(x - x_i). \qquad (4)$$

Using the standard procedure, we obtain the Hermitian collective-field Hamiltonian (Andrić and Jevicki 1983)

$$\begin{aligned} H = &\frac{1}{2} \int \mathrm{d}x\, \partial_x \frac{\delta}{\delta\rho(x)} \rho(x) \partial_x \frac{\delta}{\delta\rho(x)} + \frac{\pi^2\lambda^2}{6} \int \mathrm{d}x\, \rho^3(x) \\ &+ \frac{(\lambda-1)^2}{8} \int \frac{(\partial_x \rho(x))^2}{\rho(x)} \mathrm{d}x + \frac{\lambda(\lambda-1)}{2} \int \mathrm{d}x\, \mathrm{d}y \frac{\partial_x \rho(x) \rho(y)}{x-y} \\ &+ \frac{\omega^2}{8} \int \mathrm{d}x\, \mathrm{d}y\, \rho(x)(x-y)^2 \rho(y) - \frac{1}{4} \int \mathrm{d}x \frac{\delta\omega(x)}{\delta\rho(x)} \end{aligned} \qquad (5)$$

with

$$\omega(x) = (\lambda - 1)\partial_x^2 \rho(x) + 2\lambda \partial_x \left(\rho(x) ⨍ \frac{\rho(y)\, \mathrm{d}y}{x-y} \right). \qquad (6)$$

The last term in the Hamiltonian is divergent and, as will be shown later, it is a counter-term.

Let us show that the same kinematic part of the Hamiltonian arises in the singlet sector of the one-matrix model defined by the Lagrangian (Brezin *et al* 1978)

$$\mathscr{L} = \tfrac{1}{2} \mathrm{Tr}\, \dot{M}^2 - \mathrm{Tr}\, V(M) \qquad (7)$$

where M is the $N \times N$ matrix. We shall consider three cases: (i) the $M = R$ real symmetric matrix, (ii) the $M = H$ complex Hermitian matrix and (iii) the $M = Q$ quaternionic Hermitian self-dual matrix. The corresponding global symmetry of the Lagrangian is $\mathrm{O}(N)$, $\mathrm{U}(N)$ and $\mathrm{Sp}(N)$, respectively.

We reformulate the matrix model (7) in terms of the collective field. Let us first formulate the corresponding Hamiltonian for real symmetric matrices R. There are $\frac{1}{2}(N^2 + N)$ independent elements R_{ij} (we choose $i \leqslant j$). The kinetic part of the Hamiltonian is then proportional to the $\mathrm{O}(N)$ invariant Laplace–Beltrami operator in the space of matrices

$$\Delta_R = \sum_i \frac{\partial^2}{\partial R_{ii}^2} + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial R_{ij}^2}$$

and the total Hamiltonian is

$$H = -\tfrac{1}{2}\Delta_R + \mathrm{Tr}\, V(R).$$

Let us express the Laplace-Beltrami operator operating on the singlet sector in terms of the collective field

$$\rho(x) = \mathrm{Tr}\,\delta(x-R) = \sum_{i=1}^{N} \delta(x-x_i)$$

where x_i are eigenvalues of R. Using the chain rule for derivatives, we obtain

$$-\tfrac{1}{2}\Delta_R = \frac{1}{2}\int \mathrm{d}x\,\omega(x, R)\frac{\delta}{\delta\rho(x)} - \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\Omega(x, y; R)\frac{\delta}{\delta\rho(x)}\frac{\delta}{\delta\rho(x)}$$

where we have used the notation $\omega(x, R) = -\Delta_R\rho(x)$ and

$$\Omega(x, y; R) = \sum_i \frac{\partial\rho(x)}{\partial R_{ii}}\frac{\partial\rho(y)}{\partial R_{ii}} + \frac{1}{2}\sum_{i<j}\frac{\partial\rho(x)}{\partial R_{ij}}\frac{\partial\rho(y)}{\partial R_{ij}}.$$

In analogy with the procedure for case (iii) (Andrić and Jevicki 1983), we can express ω and Ω in terms of $\rho(x)$. All three cases can be condensed into one expression which turns out to be the same as the kinetic part in (5) with Ω as in (6). The $\mathrm{O}(N)$, $\mathrm{U}(N)$ and $\mathrm{Sp}(N)$ invariant models are obtained for $\lambda = 0, 1, 2$, respectively.

3. 1/N corrections

For the ground-state wavefunctional we take the Gaussian ansatz (Cornwall *et al* 1974)

$$\Phi(\rho) = \exp\left(-\frac{1}{4}\int \mathrm{d}x\,\mathrm{d}y(\rho(x)-\rho_0(x))G^{-1}(x, y)(\rho(y)-\rho_0(y))\right). \tag{8}$$

Here we have introduced two functions which we shall determine using the variational approach—the ground-state density $\rho_0(x)$ and the density correlations $G(x, y)$.

The ground-state energy is given by

$$E = \langle H\rangle = \left(\int \mathscr{D}\rho\,\Phi^*[\rho]H\Phi(\rho)\right)\left(\int \mathscr{D}\rho\,\Phi^*(\rho)\Phi(\rho)\right)^{-1} \tag{9}$$

and from (8) and (9) we have

$$\langle\rho(x)\rangle = \rho_0(x) \tag{10a}$$

$$\langle\rho(x)\rho(y)\rangle = \rho_0(x)\rho_0(y) + G(x, y) \tag{10b}$$

$$\int \mathrm{d}y\,G(x, y)G^{-1}(y, z) = \delta(x-z). \tag{10c}$$

In addition to these relations we must impose the condition that we are working in the N-particle sector. From (10a) we obtain

$$\int \mathrm{d}x\,\rho_0(x) = N \tag{11a}$$

and from (10b) we obtain

$$\int \mathrm{d}x\,G(x, y) = 0. \tag{11b}$$

The energy functional (9) expressed in terms of $\rho_0(x)$ and G is

$$E=\frac{\pi^2\lambda^2}{6}\int\rho_0^3(x)\,\mathrm{d}x+\frac{\lambda(\lambda-1)}{2}⨍\mathrm{d}x\,\mathrm{d}y\frac{\partial_x\rho_0(x)\rho_0(y)}{x-y}+\frac{\lambda(\lambda-1)}{2}⨍\mathrm{d}x\,\mathrm{d}y\frac{\partial_xG(x,y)}{x-y}$$
$$+\frac{(\lambda-1)^2}{8}\int\frac{(\partial_x\rho_0(x))^2}{\rho_0}\mathrm{d}x-\frac{\lambda-1}{4}\int\mathrm{d}x\,\partial_x^2\delta(x-y)\Big|_{y=x}$$
$$-\frac{\lambda}{2}\int\mathrm{d}x\,\rho_0(x)\partial_x\frac{P}{x-y}\Big|_{y=x}$$
$$+\frac{1}{4}\int\mathrm{d}x\,\rho_0(x)\partial_x\partial_yG^{-1}(x,y)\Big|_{y=x}$$
$$+\frac{(\lambda-1)^2}{8}\int\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v\,G(u,v)\rho_0(x)\partial_x\left(\frac{\delta(x-u)}{\rho_0(x)}\right)\partial_x\left(\frac{\delta(x-v)}{\rho_0(x)}\right)$$
$$+\frac{\omega^2}{8}\int\int\mathrm{d}x\,\mathrm{d}y\,\rho(x)(x-y)^2\rho(y) \tag{12}$$

where P/x means the principal value integral. Minimising the energy functional with respect to $\rho_0(x)$ and $G(x,y)$, we obtain two coupled non-linear equations:

$$\frac{\lambda^2\pi^2}{2}\rho_0^2(x)+\frac{\omega^2}{4}\int(x-y)^2\rho_0(y)\,\mathrm{d}y+\lambda(\lambda-1)⨍\mathrm{d}y\frac{\partial_y\rho_0(y)}{y-x}-\frac{(\lambda-1)^2}{8}\left[\left(\frac{\partial_x\rho_0(x)}{\rho_0(x)}\right)^2\right.$$
$$\left.-2\partial_x\left(\frac{\partial_x\rho_0(x)}{\rho_0(x)}\right)\right]+\frac{1}{4}\partial_x\partial_yG^{-1}(x,y)\Big|_{y=x}-\frac{\lambda}{4}\partial_x\frac{P}{x-y}\Big|_{y=x}=\mu_1 \tag{12a}$$

$$-\frac{1}{8}\int\tau_0(x)\partial_xG^{-1}(x,a)\partial_xG^{-1}(x,b)\,\mathrm{d}x$$
$$+\frac{\pi^2\lambda^2}{2}\delta(a-b)\rho_0(a)+\frac{\lambda(\lambda-1)}{2}\frac{1}{(a-b)^2}$$
$$+\frac{(\lambda-1)^2}{8}\int\mathrm{d}x\,\rho_0(x)\partial_x\left(\frac{\delta(x-a)}{\rho_0(x)}\right)\partial_x\left(\frac{\delta(x-b)}{\rho_0(x)}\right)+\frac{\omega^2}{8}(a-b)^2=\mu_2 \tag{12b}$$

where μ_1 and μ_2 are Lagrange multipliers which ensure the conditions (11a) and (11b), respectively. From these two equations we should determine $\rho_0(x)$ and $G^{-1}(x,y)$ subjected to the constraints (11a) and (11b). In order to eliminate the singular terms in equations (12), we shall assume the ansatz

$$G^{-1}(x,y)=-2N\ln|x-y|-(\lambda-1)\frac{\delta(x-y)}{\rho_0(x)}+K(x,y) \tag{13}$$

where K is a singularity-free part of $G^{-1}(x,y)$. This ansatz will automatically eliminate the divergences (in the leading order in N) which also appear in the energy functional (12). Using (12a) and (12b), one can write the ground-state energy as

$$E=\frac{\pi^2\lambda^2}{6}\int\rho_0^3(x)\,\mathrm{d}x+\frac{\omega^2}{8}\int\rho_0(x)(x-y)^2\rho_0(y)+\frac{(\lambda-1)^2}{8}\int\frac{(\partial_x\rho_0(x))^2}{\rho_0(x)}\mathrm{d}x$$
$$+\frac{\lambda(\lambda-1)}{2}\int\mathrm{d}x\,\mathrm{d}y\frac{\partial_x\rho_0(x)\rho_0(y)}{x-y}+\frac{1}{4}\int\mathrm{d}x\,\rho_0(x)\partial_x\partial_yG^{-1}(x,y)\Big|_{y=x}$$
$$-\frac{\lambda}{2}\mathrm{d}x\,\rho_0(x)\partial_x\frac{P}{x-y}\Big|_{y=x}-\frac{\lambda-1}{4}\int\mathrm{d}x\,\partial_x^2\delta(x-y)\Big|_{y=x} \tag{14}$$

Inserting (13) in (12*a*) and (12*b*), we can reduce the problem of finding $\rho_0(x)$ to a simpler problem. If $\rho_0(x)$ satisfies the equation

$$\lambda \fint \frac{\rho_0(y)\,\mathrm{d}y}{x-y} + \frac{\lambda-1}{2}\frac{\partial_x \rho_0(x)}{\rho_0(x)} = \omega\sqrt{N/2}\,x \tag{15a}$$

and

$$K(x,y) = \frac{\omega}{4}\sqrt{2/N}\,[x^2+y^2-4xy] \tag{15b}$$

then it is the solution of the system (12*a*) and (12*b*). To prove this, we need to square equation (15*a*). Using the identity for the principal distributions (Jackiw and Strominger 1981)

$$\frac{P}{x-y}\frac{P}{x-z} + \frac{P}{y-z}\frac{P}{y-x} + \frac{P}{z-x}\frac{P}{z-y} = \pi^2\delta(x-y)\delta(x-z)$$

we obtain equation (12*a*).

From equation (15*a*) we can discuss the qualitative features of the solution in terms of the parameter λ. For λ in the range $0<\lambda<1$ and large x, the first term is vanishing and therefore $\rho_0(x)$ is vanishing like Gaussian. For $x=0$, $\rho_0(x)$ has a maximum. For $\lambda=0$, the solution is a Gaussian function and for $\lambda=1$, we have a solution defined on the compact support. For both these cases, the solution $\rho_0(x)$ can be found explicitly.

The ground-state energy, together with the leading corrections from (14), is given by

$$E = \frac{\pi^2\lambda^2}{6}\int \rho_0^3(x)\,\mathrm{d}x + \frac{\omega^2}{8}\int \rho_0(x)(x-y)^2\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{(\lambda-1)^2}{8}\int \frac{(\partial_x\rho_0(x))^2}{\rho_0(x)}\,\mathrm{d}x$$
$$+\frac{\lambda(\lambda-1)}{2}\int \frac{\partial_x\rho_0(x)\rho_0(y)}{x-y}\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{4}\int \mathrm{d}x\,\rho_0(x)\partial_x\partial_y K(x,y)\Big|_{y=x}. \tag{16}$$

In this particular case when $\rho_0(x)$ satisfies equation (15*a*), we can evaluate the integrals in (16) by using equation (15*a*) and we obtain

$$E = \frac{\omega}{2}\sqrt{N/2}\,[\lambda N(N-1)+N-1]. \tag{17}$$

This is the expression for the leading order in N and its next correction in the large-N expansion. This is also, not unexpectedly, the exact result for finite N. We can show this by using scaling arguments (Andrić *et al* 1983) owing to the scaling properties of the potential.

Now we can also determine the Schrödinger wavefunctional of N particles expressed in terms of the collective field, G^{-1} and K:

$$\psi(\rho) = \Delta\exp(-\tfrac{1}{2}\ln J(\rho))\Phi(\rho). \tag{18}$$

The first factor is an antisymmetric prefactor, the second factor is due to the Hermitisation of the Hamiltonian and the third factor is a collective vacuum wavefunctional. The Jacobian is determined by using the standard procedure when going from explicit coordinate dependence to the Hermitian collective-field treatment. The explicit structure of the Jacobian is given by

$$\ln J = (\lambda-1)\int \rho(x)\ln\rho(x)\,\mathrm{d}x + \lambda\int \mathrm{d}x\,\mathrm{d}y\,\rho(x)\ln|x-y|\rho(y).$$

Expanding the wavefunctional around the stationary point $\rho_0(x)$ up to quadratic fluctuations and using equation (15a), we obtain the wavefunctional

$$\psi(\rho)=\Delta\exp\left(-\frac{\omega}{4}\frac{1}{\sqrt{2N}}\int\rho(x)(x-y)^2\rho(y)\,dx\,dy\right). \tag{19}$$

This is also an exact result in the case with finite N (Calogero 1969). If we substitute equation (4) for finite N, we obtain the exact wavefunction for N particles.

Summarising, we can point out the role of the parameter λ. For $\lambda=0$, we have a symmetric wavefunction and the corresponding ground-state energy is identical with the ground-state energy of N bosons. For $\lambda=1$, the wavefunction is completely antisymmetric and the corresponding ground-state energy is exactly that of the system of N fermions with the harmonic-potential interaction (Levy-Leblond 1969). Here we have unified the treatment of one-dimensional bosons and fermions via the singular interaction which represents continuous perturbations to the Hamiltonian. In this context, the role of the parameter is to determine statistics. For $\lambda\neq 0$, 1, it represents intermediate statistics.

For the Sp(N) invariant matrix model ($\lambda=2$) with the interaction

$$\mathrm{Tr}\,V(M)=\frac{\omega^2}{2}\mathrm{Tr}\,M^2=\frac{\omega^2}{2}\sum_{i=1}^{N}x_i^2 \tag{20}$$

the ground-state energy, together with the corrections, can be obtained from (17) because the two-particle harmonic interaction is equivalent to the interaction (20) up to the centre-of-mass motion and rescaling. Then equation (17) becomes

$$E=\frac{\omega}{2}[\lambda N(N-1)+N]=\frac{\omega}{2}(2N^2-N)$$

which is the exact result for finite N because there are $2N^2-N$ independent variables describing quaternionic matrices.

4. Conclusion

We have developed a consistent straightforward way of treating higher terms in the $1/N$ expansion using the collective-field treatment. In the examples given above, we have also obtained exact results for finite N. In the collective-field formulation, we have shown an explicit equivalence between the singlet sector of the O(N), U(N) and Sp(N) invariant matrix models and particles with intermediate statistics.

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